The Dimensionless Gasket Width *b(c,n)* in Statistical Geometry.

1. Introduction. Statistical Geometry.

It has been found [1]-[3] that if one randomly places shapes whose areas follow a prescribed power law the result is a space-filling *non-Apollonian* fractal. Such constructions are referred to here as "statistical" geometry since they add statistical properties to the known properties of geometric shapes such as circles, squares, etc.

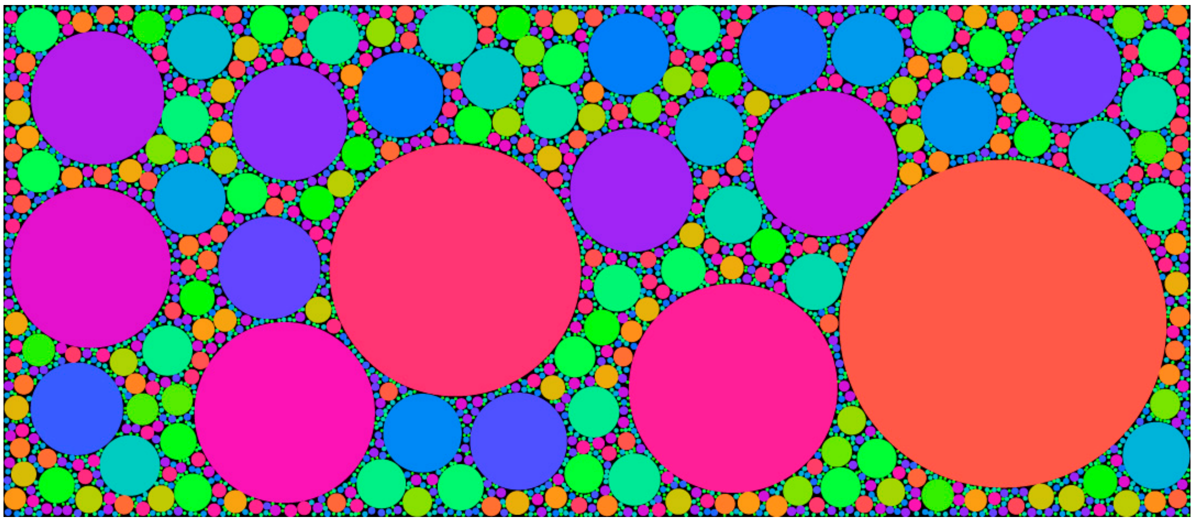

Fig. 1. An example of statistical geometry. Randomly-placed fractal circles with log-periodic color. Space-filling in the limit of infinitely many circles. Black gasket.

While the fractal distribution of shapes in statistical geometry is interesting and eye-catching, the "gasket" of leftover space may be the more interesting story. In most images of these fractals the gasket has low visibility, but it contains a lot of structure and information. A mathematical approach to describing the properties of the gasket could be very interesting. A start is made here.

2. The Gasket.

The discussion here assumes that the shapes are circles. Generalization to other shapes is straightforward.

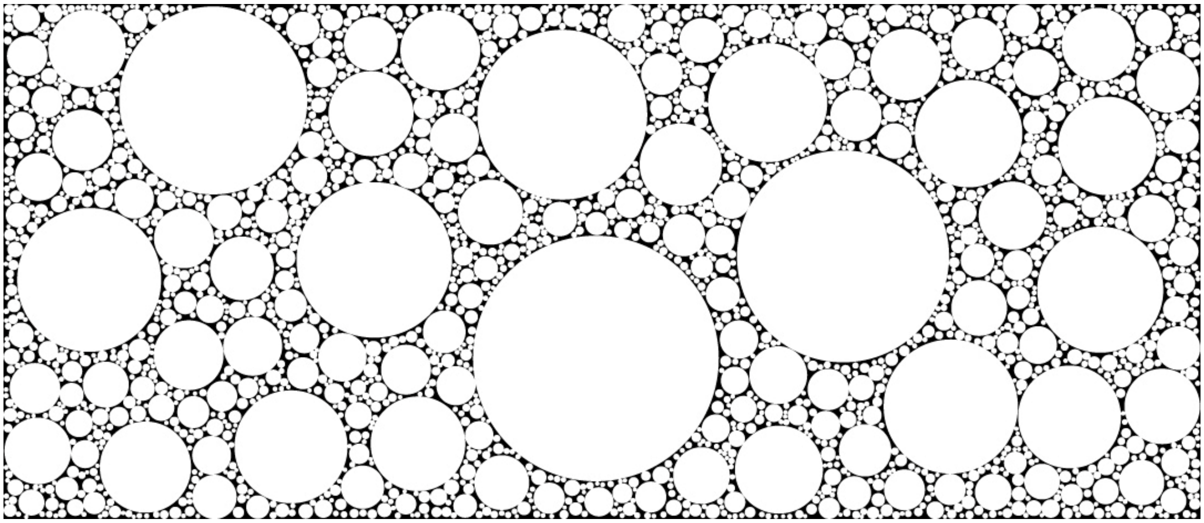

Fig. 2. A black-and-white image of a gasket with circles. c=1.32. Fill factor 87%; gasket is 13% of the area. By construction the gasket is a single connected whole.

The setup procedure is reviewed. The areas of the shapes are to follow a power law, and the resulting set is to be space-filling. This means that the n-th shape placed will have an area proportional to $1/n^c$. (We here assume that the sequence starts with $i=1$, but the result is easily generalized to the case where we start with $i=N$. In that case the Riemann zeta function is replaced by the Hurwitz zeta function.) We can find the (infinite) sum S of these terms as

$$S = \sum_{i=1}^{\infty} \frac{1}{i^c} = \zeta(c) \qquad (1)$$

where $\zeta(c)$ is the Riemann zeta function. We will assume that the shapes are to be placed in an area A. The areas A_i of the shapes that will result in a space-filling set are then:

$$A_i = \frac{A}{\zeta(c) i^c} \qquad (2)$$

where i = 1, 2, … . For a circle the corresponding perimeter p_i and radius r_i are then

$$p_i = 2\sqrt{\pi}\sqrt{A_i} = 2\sqrt{\pi}\sqrt{\frac{A}{\zeta(c) i^c}} = 2\sqrt{\pi}\sqrt{\frac{A}{\zeta(c)}}\frac{1}{i^{c/2}} \qquad (3)$$

$$r_i = \sqrt{\frac{A_i}{\pi}} = \sqrt{\frac{A}{\pi\zeta(c) i^c}} = \frac{1}{\sqrt{\pi}}\sqrt{\frac{A}{\zeta(c)}}\frac{1}{i^{c/2}}$$

We can make a crude estimate of the average gasket width w_g after n shapes have been placed by

$$w_g(n) = \frac{gasket\ area\ after\ n\ placements}{gasket\ perimeter\ after\ n\ placements}$$

$$= \frac{A - \frac{A}{\zeta(c)}\sum_{i=1}^{n}\frac{1}{i^c}}{2\sqrt{\pi}\sqrt{\frac{A}{\zeta(c)}}\sum_{i=1}^{n}\frac{1}{i^{c/2}}} \qquad (4)$$

The "yardstick" for measuring such a gasket width is the diameter $2r_{n+1}$ of circle $n+1$ -- the next one needing to be placed. Is it wide or narrow compared to the spaced needed? We express this mathematically as the dimensionless average width $b(c,n)$

$$b(c,n) = \frac{w_g}{2r_{n+1}} = \frac{1}{2}\frac{A - \frac{A}{\zeta(c)}\sum_{i=1}^{n}\frac{1}{i^c}}{\frac{1}{\sqrt{\pi}}\sqrt{\frac{A}{\zeta(c)}}\frac{1}{(n+1)^{c/2}}2\sqrt{\pi}\sqrt{\frac{A}{\zeta(c)}}\sum_{i=1}^{n}\frac{1}{i^{c/2}}} \qquad (5)$$

If we collect and cancel terms we find

$$b(c,n) = \frac{w_g}{2r_{n+1}} = \frac{1}{4}\frac{\zeta(c) - \sum_{i=1}^{n}\frac{1}{i^c}}{\frac{1}{(n+1)^{c/2}}\sum_{i=1}^{n}\frac{1}{i^{c/2}}} = \frac{1}{4}\frac{\sum_{i=n+1}^{\infty}\frac{1}{i^c}}{\frac{1}{(n+1)^{c/2}}\sum_{i=1}^{n}\frac{1}{i^{c/2}}} \qquad (6)$$

The reader will notice that this has nothing to do with randomness. $b(c,n)$ is an ordinary deterministic function. It requires calculation of various partial sums of the series for $\zeta(c)$ and $\zeta(c/2)$.

It is not at all obvious what the limiting behavior(s) of this complicated expression will be versus n and c. In the following table it is believed that the calculations are accurate to the full 6 digits given.

3. Numerical Results for Circles.

This table assumes N=1, i.e. the sequence starts with the 1 term.

$b(c,n)$	n=1000	n=4000	n=16000	n=64000	n=256000	Average trials per placement at 90% fill
c=1.20	0.526070	0.514602	0.508272	0.504715	0.502696	~49
c=1.25	0.398863	0.388794	0.383072	0.379756	0.377813	~98
c=1.30	0.314371	0.305204	0.299846	0.296646	0.294711	~315
c=1.35	0.254314	0.245770	0.240639	0.237485	0.235518	~920

The dependence on *n* is quite weak; *b* drops very gradually with increasing *n*. This helps to explain why the algorithm works. The relative amount of space available is nearly constant as *n* varies -- the available space shrinks at the same rate as the size of the circles to be placed.

There is a much stronger dependence on *c*. A small change in *c* produces a sizable change in *b*, i.e., the gasket gets much "thinner" with large *c*. It should be kept in mind that *b* is the *average* gasket width, but that the width has fluctuations from one place in the pattern to another and that there are places with a larger than average width -- as shown by the fact that the algorithm easily places new circles even when the average gasket width is only half a diameter (*b*=.5).

The last column shows the average number of trials needed to make a new placement with the area 90% filled. This is necessarily a *statistical estimate*, with errors that have not been determined. It can be seen that this rises steeply (almost exponentially) with increasing *c*. The reciprocal of this number can be viewed as the "placement probability" under the given conditions and is a *statistical property of the circle*. Such data, if properly interpreted, can give a statistical description of the lacy fractal gasket.

This is about as far as one can go with the average gasket width w_g. An interesting project for circles would be to create software that would identify adjacent triplets of circles in a set of data such as that for Fig. 1 or Fig. 2 and then use Apollonius's method to find the largest (tangent) circle which will fit in this space. A histogram of the circle radii thus found would give an interesting statistical description of the gasket. It would presumably have a largest radius, i.e., the underlying distribution has a hard upper limit.

4. References.

[1] Statistical geometry article on the web site john-art.com.

[2] John Shier, "Hyperseeing", Summer 2011 issue, pp. 131-140, published by ISAMA. Available at the web site john-art.com.

[3] "Statistical Geometry", John Shier, July 2011. A self-published fractal art picture book available at lulu.com.

John Shier

July 2011