Holes in Circle Fractals. Statistics

A statistical study has been made of the "holes" in circle "statistical geometry" fractal patterns [1]-[3] (see Appendix A). Figure 1 shows an example of a computer run made for this purpose.

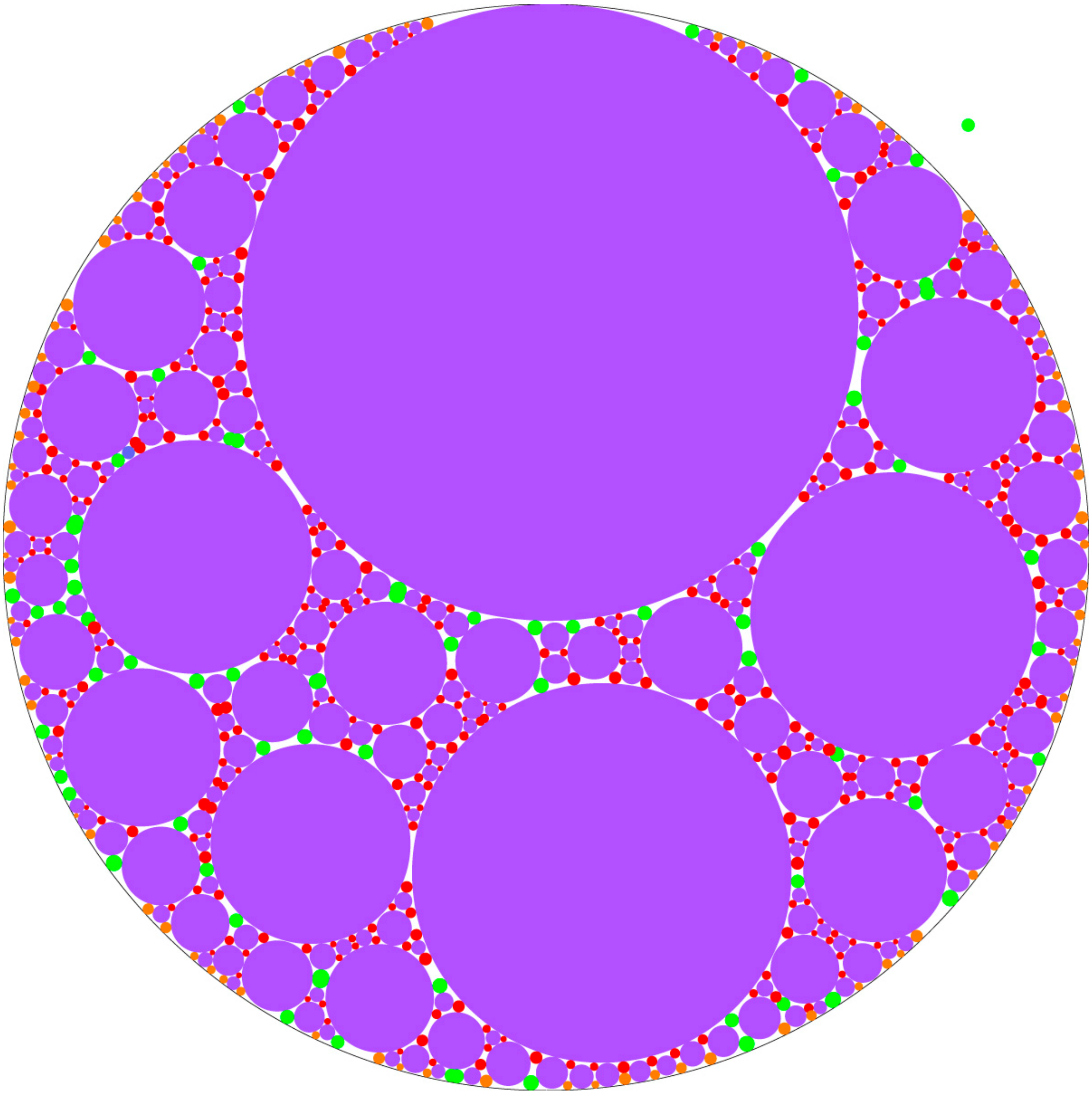

Fig. 1. A circles-in-circle fractal pattern. The purple circles are the 252 circles placed by the algorithm. (c=1.4, N=1, 91.2% fill). The small circles inside the pattern are the 502 "holes" where the next circle can go. The next circle is drawn (in green) at proper scale up and to the right of the pattern as a comparison. The green holes are those large enough for the next circle to fit. The red holes are "internal" holes, not tangent to the bounding circle. The orange holes are those which are tangent to the bounding circle. The dark blue circle is the smallest placed circle.

1. Introduction. Methods.

One of the interesting questions about the statistical geometry algorithm is the matter of "available places for the next shape" and the statistical distribution of their sizes. For circles these are simply the tangent "Apollonian" circles which fit within the available space between triplets of placed circles (Fig. 1).

The holes found evidently include the largest available spaces for a next placement, but it can be seen from Fig. 1 that there is available space for yet-smaller circles which is not included by this search process.

One can think of the process of placing circles as a progression in which the placement of a new circle fills one hole and creates three new ones, for a net gain of 2. It is found that in fact the number of holes is 2(n-1) when n circles have been placed.

The object is to find the probability distribution function for the hole diameters. Statistical accuracy has been improved by making many runs and combining the results.

The construction of a circle tangent to three given circles when the three are not mutually tangent[1] is not a simple problem and as far as I can determine there is no algebraic solution to the three nonlinear equations which must be solved. Here the solution has been found by developing initial estimates of the position and radius of the "hole" circle and using the Newton-Raphson[2] method to refine and improve the values. In practice this proved to be quite rapidly convergent and the estimated accuracy of the hole x, y, and r is ±.00001 inches in a 10x10 inch drawing area. The iterative computations were done in double precision arithmetic.

2. The Distribution of Hole Radii.

The hole size distribution has a bearing on the question "Will the algorithm stop?"

If the available places are so few or so small that random search is unlikely to find them the search will take a long time. A calculation of this kind allows one to compute the probability for placement of the next circle, and to estimate the average number of trials needed, which in turn can be compared to actual run data.

Repeated runs were made with c=1.45, N=1, and 91.2% fill (252 circles) and also with 84.4% fill (126 circles). These parameters imply a tightly-packed highly-correlated fractal where the number of big-enough holes is limited at any step. Figure 2 shows the results in the form of a histogram.

It should be kept in mind that this is an average distribution, and that the number of "big enough" holes will vary from run to run. There may be a finite probability that under some conditions the process

[1] For the case of 3 *mutually tangent* circles "Soddy's formula" provides an elegant result.

[2] The one-dimensional version of this iterative method for solving nonlinear equations is covered in most introductory calculus texts as "Newton's method". Many texts and internet references cover the n-dimensional case, often under the heading of "nonlinear regression" (as applied to curve fitting). The complex-number case is used in generating the "Mandelbrot set" and other Julia sets.

arrives at a given point and there are no available holes due to the fluctuations of the random numbers. Experience suggests that this is most likely to happen when c is large and n is small, and examples of this form of stoppage seem to occur fairly often when c ≅ 1.5 (near its upper limit). Stoppage for "no hole" with high c usually occurs in the first few placements.

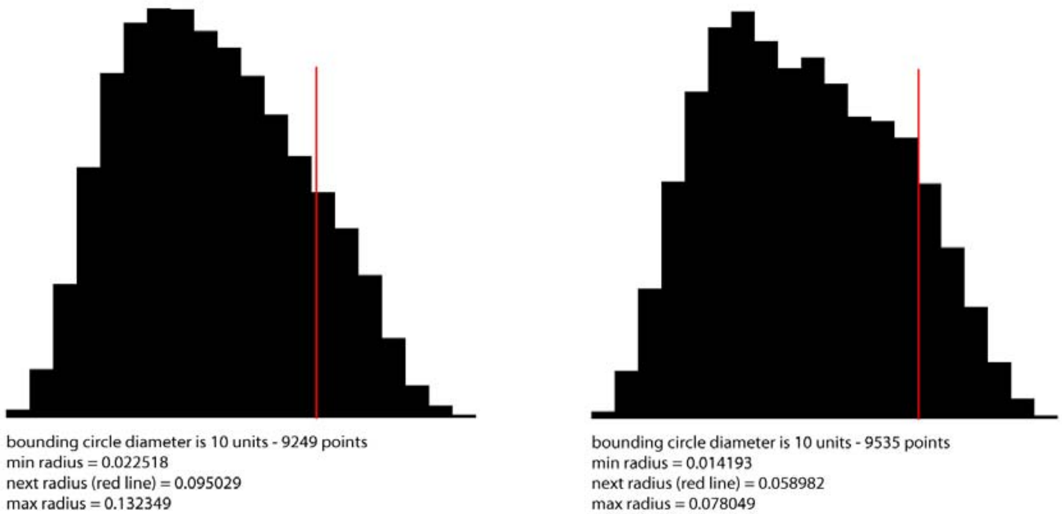

Fig. 2. Histograms of the tangent hole radii with the c and N values stated, for a circle fractal as in Fig. 1. The vertical red line shows the radius of the next circle needing to be placed. In all the cases studied there were substantial numbers of holes with radii big enough to accommodate the next circle. The left histogram is for 126 circles placed (84.4% fill), while the right one is for 252 circles placed (91.2% fill).

3. Discussion of the Results.

The hole-radius probability distribution function[3] for 126 placed circles (left) looks like a typical "skewed" p.d.f. that one might find in statistics textbooks.

The hole-radius probability distribution function for 252 placed circles (right) is somewhat different and does not appear (within sampling error) to be any of the standard distributions found in textbooks. It tails off rapidly for small r and for r values larger than the "next radius". In the central region it appears to have a slow and almost linear drop. The tail for large r appears to be "exponential-like" (with statistical accuracy). The large-r tail likely extends beyond the range[4] of the data shown here.

In both cases a substantial fraction of the holes are larger than the next to-be-placed circle.

[3] This is, of course, only sampled data. The "true" continuous p.d.f. can be defined as the average over all possible configurations of n randomly-chosen nonoverlapping circle positions with the given c,N. In statistical physics such an average is called an "ensemble" average (over all "states").

[4] The largest hole radii will be quite rare and will correspond to the (unlikely) case where the circles are nearly tangent leaving room for a large hole somewhere. A very large data sample would be needed to establish the character of this tail accurately.

If one thinks of a continuous process of adding circles, the filling of existing holes removes area from the right-hand side of the histogram (to the right of the red line) while it creates additional area to the left of the red line.

How do these distributions relate to the dimensionless gasket width b(c,n)? It is stated [4] that b(c,n) is the "average" width[5] of the gasket. Here we can define a different dimensionless width of the gasket -- the width of the p.d.f. for r. For comparison purposes I take this to be the maximum r minus the minimum r.

For 126 circles placed (distribution width)/(r of next circle) = 1.08 (b(c,n) = .197)

For 252 circles placed (distribution width)/(r of next circle) = 1.15 (b(c,n) = .188)

This suggests that the dimensionless hole-radius distribution width is about 5 times the value of b(c,n). A more precise comparison of this data would require a more precise definition of just how the distribution width is to be determined from the data in Fig. 2.

The dimensionless ratio (maximum r of the distribution)/(r of next circle) appears to track rather closely with b(c,n); it has a value around 1.39 for 126 circles placed. This is further evidence that the process does not stop, especially for large numbers of placed circles where the hole distribution becomes quite dense.

4. References.

[1] Statistical geometry article on the web site john-art.com.

[2] John Shier, "Hyperseeing", Summer 2011 issue, pp. 131-140, published by ISAMA. Available at the web site john-art.com.

[3] "Statistical Geometry", John Shier, July 2011. A self-published fractal art picture book available at lulu.com.

[4] Unpublished article "The Dimensionless Gasket Width b(c,n) in Statistical Geometry", John Shier, July 2011. It can be found at the web site john-art.com.

John Shier

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Appendix A. The Statistical Geometry Algorithm.

It has been found [1]-[3] that it is possible to create fractal patterns of a wide variety of geometric shapes by the following algorithm:

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[5] The definition of b(c,n) is (total gasket area)/(total gasket perimeter)(diameter of next circle to be placed). One needs to be careful in thinking about this dimensionless quantity as an average because the gasket is neither a two-dimensional nor a one-dimensional object but a fractal.

1. Create a sequence of areas A_i equal to $\frac{1}{N^c}, \frac{1}{(N+1)^c}, \frac{1}{(N+2)^c}, \frac{1}{(N+3)^c}, \ldots$. Choose an area (square, rectangle, circle, …) A to be filled.

2. Sum the areas A_i to infinity, using a zeta[6] function

$$z(c,N) = \sum_{i=0}^{\infty} \frac{1}{(N+i)^c}$$

3. Define a new set of areas S_i by $S_i = \frac{A}{z(c,N)(N+i)^c}$. It will be seen that the sum of all these redefined areas is just A.

4. Let $i = 0$. Place a shape having area S_0 in the area A at a random position x,y such that it falls entirely within area A. Increment i. This is the "initial placement".

5. Place a shape having area S_i entirely within A at a random position x,y such that it falls entirely within A. If this shape overlaps with any previously-placed shape repeat step 5. This is a "trial".

6. If this shape does not overlap with any previously-placed shape put x,y and the shape dimensions in the "placed shapes" data base, increment i, and go to step 5. This is a "placement".

7. Stop when i reaches a set number, percentage filled area reaches a set value, or other.

One will note that the dimensions of the shapes are nowhere specified. They are calculated from the areas. A very wide variety of shapes have been found to be "fractalizable" in this way.

The parameters c and N can have a variety of values. The parameter c is often in the range 1.2-1.4 with a largest usable value around 1.51. N can be 1 or larger, and need not be an integer.

By construction the result is a space-filling random fractal -- if the process never halts. Available evidence [1]-[3] says that it does not. The power law area sequence ensures that it has the fractal "statistical self-similarity" (scale-free) property. And the random search ensures that no two circles will ever touch, so that the "gasket" is a single continuous object.

[6] The definitions of the Riemann and Hurwitz zeta functions can be found in Wikipedia.